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Compatible stem taper and stem volume functions for oak (*Quercus robur* L and *Q petraea* (Matt) Liebl) in Denmark

MJ Tarp-Johansen [1], JP Skovsgaard [1]*, SF Madsen [1], VK Johannsen [1], I Skovgaard [2]

[1] *Danish Forest and Landscape Research Institute, Department of Forestry, Hørsholm Kongevej 11, DK-2970 Hørsholm;*
[2] *Royal Veterinary and Agricultural University, Department of Mathematics and Physics, Thorvaldsensvej 40, DK-1871 Frederiksberg C, Denmark*

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Summary – In this paper we develop compatible stem taper and stem volume functions for oak (*Quercus robur* L and *Q petraea* (Matt) Liebl) in Denmark. In the compatible system of stem taper and volume functions the solid of revolution of the stem taper equation equals the volume according to the total stem volume function. The stem volume is explicitly expressed in the stem taper function. Thus, it is possible to adjust estimates of the stem taper curve to a specific stand volume level (and thereby accounting for the effect of silvicultural practice on stem taper). The accuracy of predictions from the stem volume function for oak is comparable to similar functions for three broadleaved species and six conifers. Comparable stem taper functions for other broadleaved species are not available, but compared to conifers the stem taper function for oak performs relatively well considering the substantial number of forked oak trees.

stem taper / stem volume / oak / *Quercus robur* / *Quercus petraea*

Résumé – Système compatible d'équations de défilement et de tarif de cubage pour les chênes au Danemark. Dans cet article, nous développons des équations de défilement et des tarifs de cubage compatibles pour les chênes (*Quercus robur* L et *Q petraea* (Matt) Liebl) au Danemark. Dans ce système cohérent, le solide de révolution engendré par l'équation de défilement a le même volume que celui fourni par le tarif de cubage de la tige. Le volume du tarif de cubage est exprimé explicitement à partir de l'équation de défilement. Ainsi, il est possible d'ajuster les estimateurs de la forme de défilement à un niveau local du tarif de cubage (et, de là, prendre en compte les influences

* Correspondance and reprints
Tel: (45) 45 76 32 00; fax: (45) 45 76 32 33; e-mail: jps@fsl.dk

des pratiques sylvicoles dans le défilement de la tige). La précision des prévisions faites à partir du tarif de cubage est du même ordre de grandeur que des fonctions similaires pour trois espèces feuillues et six de conifères. Des équations de défilement pour d'autres espèces feuillues ne sont pas disponibles, mais comparé au résultat pour les conifères, l'ajustement de la fonction de défilement pour les chênes est satisfaisant, étant donné notamment le nombre élevé de chênes fourchus.

défilement / volume tige / chêne / *Quercus robur* / *Quercus petraea*

INTRODUCTION

Compatible stem taper and stem volume functions for commercial tree species are very useful and flexible tools for both forestry practice and forest research. These functions provide estimates of stem diameter at any height and estimates of total stem volume as well as merchantable volume at any stem diameter along the trunk. Such functions may be used on their own or in combination with growth models to simulate the distribution of tree sizes and volumes.

In the compatible system of stem taper and volume functions, the solid of revolution of the stem taper equation equals the volume according to the total stem volume function. The stem volume is explicitly expressed in the stem taper function. Thus, it is possible to adjust estimates of the stem taper curve to a specific stand volume level (and thereby accounting for the effect of silvicultural practice on stem taper).

The concept of compatibility of stem taper and stem volume functions is probably as old as the idea of modelling these proporties. The concept seems to have been formally introduced by Demaerschalk (1972, 1973) for total volume, and subsequently refined by Burkhart (1977) and Clutter (1980) to include functions for merchantable volume. Compatible systems may be taper-based or volume-based (Munro and Demaerschalk, 1974), depending on which function is derived first.

In this paper we develop compatible stem taper and stem volume functions for oak (*Quercus robur* L and *Q petraea* (Matt) Liebl) in Denmark. The system is volume-based and the approach follows the model concept previously developed for conifers (Madsen, 1985, 1987; Madsen and Heusèrr, 1993; see also Goulding and Murray, 1976). The further development, compared to the traditional compatible system, includes two refinements (introduced by Madsen, 1985): 1) additional restrictions on the stem taper model in order to improve model performance, and 2) variables that simultaneously account for butt swell and taper in the uppermost part of the stem.

MATERIAL

The material comprises 1131 sample trees from a total of 38 plots in long-term experiments, conducted by the Danish Forest and Landscape Research Institute. Data collection took place between 1902 and 1977. Geographically the plots are unevenly distributed (fig 1), with two thirds of the plots located on the island of Zealand. Moreover, one particular forest district (Bregentved) is represented by 314 trees or approximately 28% of the material, and one particular plot (sample plot QA, Stenderup) contributes as many as 90 trees. Unfortunately, the data include only one 'genuine' thinning experiment (thinning grades B, C and D) and only for a very narrow range of ages (21-27 years).

Summary statistics are given in table I. Stands are thinned from below, and most sample trees chosen among thinned trees. Trees were sampled to represent the variation, but in young stands with a tendency to favour 'average' crop trees. Figure 2 shows the quadratic mean diameter and the diameter range of sample trees compared to the mean and range of remaining

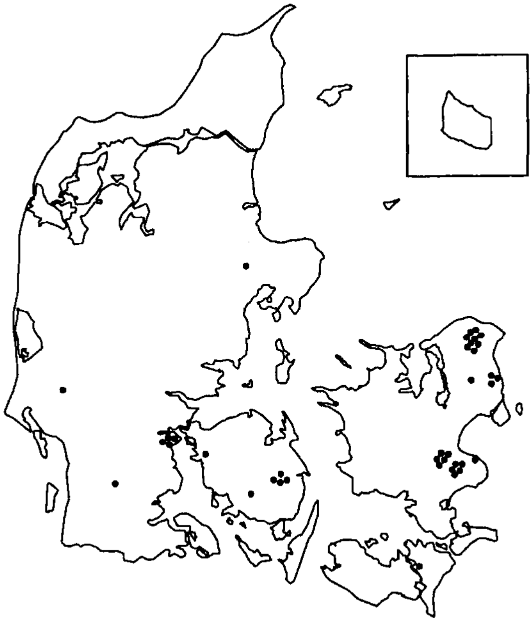


Fig 1. Geographical distribution of 38 oak plots in Denmark

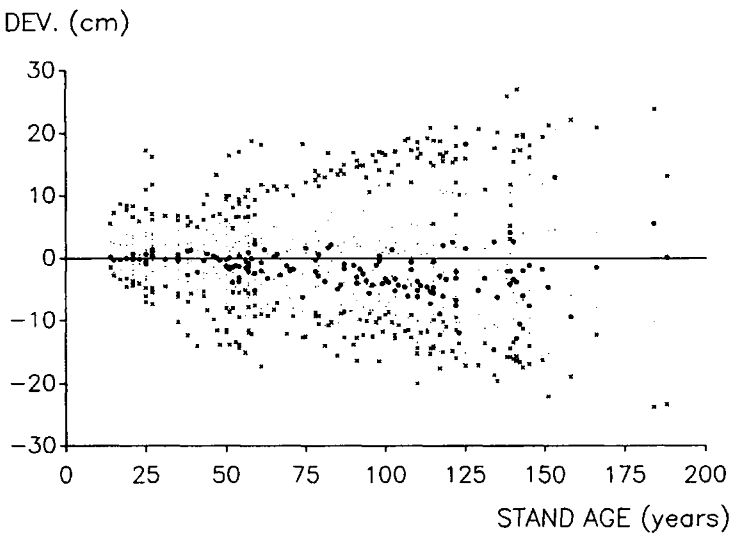


Fig 2. Maximum, quadratic mean and minimum diameter of sample trees compared to remaining crop trees. For any stand age, five values are shown. Beginning from the top they are: × = $d_{\text{max,stand}} - D_{g\text{,stand}}$; · = $d_{\text{max,sample}} - D_{g\text{,stand}}$; • = $D_{g\text{,sample}} - D_{g\text{,stand}}$; · = $d_{\text{min,sample}} - D_{g\text{,stand}}$; × = $d_{\text{min,stand}} - D_{g\text{,stand}}$. Pairwise, the values of ×'s show the diameter range for the remaining crop trees, and the •'s show the diameter for sample trees. Symbols and subscripts in formulae: d = diameter at breast height; D_g = quadratic mean diameter; max = maximum, min = minimum, stand = remaining crop trees, sample = sample trees.

Table I. Summary statistics of oak material.

Variable		*Mean*	*St dev*	*Min*	*Max*
Diameter at breast height	$d_{1.30}$ (cm)	29.7	19.2	0.8	111.4
Tree height	h (m)	18.4	6.7	2.0	32.4
Stem volume	v (m^3)	1.1489	1.6285	0.0002	11.1704
Stand mean diameter	D_g (cm)	31.4	20.0	2.4	87.5
Stand height	H_g (m)	18.8	6.9	3.6	29.6
Age	T (years)	72	44	14	188

crop trees. The figure illustrates that, with increasing age, the size of the sample trees decreases relative to the size of the remaining crop trees, reflecting the combination of thinning and sampling practices. A total of 215 sample trees (19%) were forked, mostly with the fork base being located at a height between one and two thirds of the tree height. Further details have been reported in a previous paper on functions for total tree volume (Madsen, 1987).

All sample trees were felled and the logs measured according to the so-called relative method (Oppermann and Prytz, 1892; see also Madsen, 1987). With this method, the log is divided into two main sections, one above and one below 1.30 m above ground level. The section above is subdivided into ten subsections of equal length $(h - 1.30)/10$, hence the name of the method. The section below 1.30 m is subdivided into four subsections of equal length (32.5 cm). Subsections above 1.30 m are calipered cross-wise at the end, subsections below 1.30 m at the middle (at heights 0.16, 0.49, 0.81 and 1.14 m). Stem diameter at any height is calculated as the average of the two perpendicular measurements. All measurements are taken outside the bark.

Above 1.30 m subsection volumes are calculated according to Smalian's formula, below 1.30 m according to Huber's formula. Stem diameters for forked trees are transformed into a combined diameter measure corresponding to the joint cross-sectional area of the forks. To minimize bias in stem taper functions, our experience suggests that the diameter at 0.16 m height be set equal to the diameter at 0.49 m. Diameter at the tree top is set equal to zero. Stump height, $\hat{h}_s$, is estimated by the empirically derived formula: $\hat{h}_s = 0.12 + d_{1.30}/4$ (Madsen, 1987), where $d_{1.30}$ is the diameter at breast height. Thus, 'true' stem volume is calculated as

$$v = \frac{\pi}{4}\Big((2 \cdot 0.325 - \hat{h}_s)d_{0.49}^2 + 0.325(d_{0.81}^2 + d_{1.14}^2) + \frac{h - 1.3}{10}(\frac{1}{2}d_{1.30}^2 + d_{0.1}^2 + d_{0.2}^2 + \ldots + d_{0.9}^2)\Big) \quad [1]$$

where v is total stem volume excluding stump, $\hat{h}_s$ denotes estimated stump height, h is total height from ground to tip of tree, $d_{0.49}$, $d_{0.81}$, $d_{1.14}$ and $d_{1.30}$ are the diameters at a height of 0.49, 0.81, 1.14 and 1.30 m, respectively, and $d_{0.1}, d_{0.2}, \ldots, d_{0.9}$ are the diameters at the relative heights $n(h - 1.30) + 1.30$ for $n = 0.1, 0.2, \ldots, 0.9$. The calculation of v in cubic metres implies all diameter and height measures to be inserted in equation [1] in metres.

METHODS

Stem volume function

The chosen stem volume model originates from the classic stem taper function (Riniker, 1873; see for example Philip, 1994)

$$d_l^2 = 4p(h - l)^r \quad [2]$$

where d_l designates the diameter at a given height l (above ground), h is the total tree height, and the numerical constants p and r define the rate of taper and the shape of the solid, respectively ($p > 0$, $r \geq 0$); r is called the form exponent. The solid of revolution, ie, the stem volume, is thus

$$\begin{aligned} v &= \pi \int_0^h p(h-l)^r \mathrm{d}l \\ &= \frac{\pi p}{r+1} h^{r+1} \qquad [3] \\ &= \frac{\pi}{4} d_{1.30}^2 \frac{1}{r+1} h \left(\frac{h}{h-1.30} \right)^r \qquad [4] \end{aligned}$$

and by logarithmic transformation (base e) of equation [4], a linear model results:

$$\begin{aligned} \ln(v) &= \ln\left(\frac{\pi}{4}\right) + 2\ln(d_{1.30}) \\ &\quad - \ln(r+1) + \ln(h) \qquad [5] \\ &\quad + r \cdot \ln\left(\frac{h}{h-1.3}\right) \end{aligned}$$

By adding stand parameters to allow for temporal changes within a stand (plot), by making proper assumptions about $\ln(r+1)$ and $r \cdot \ln(h/[h-1.30])$ (Madsen, 1987; cf Kunze, 1881) and by defining a level, A, specific to the stand (plot), the logarithmic stem volume model becomes

$$Y_{ij} = A_i + \sum_k b_k x_{ijk} + e_{ij} \qquad [6]$$

where i denotes stand (plot) number; j denotes tree number (in stand i); k denotes predictor number; $Y_{ij} = \ln(v_{ij})$; v_{ij} is the stem volume (m^3) of tree no ij; A_i is the stand (plot) specific level (assumed $N(\mu, \sigma_A^2)$ and mutually independent); b_k denotes the k'th coefficient; x_{ijk} or x_k in abbreviated notation, denotes the k'th predictor variable (full list in *Appendix* 1), where in particular: $x_1 = \ln(d_{1.30})$, $x_2 = \ln(h)$, $x_4 = \ln(h/[h-1.30])$, $x_{25} = D_g^2$; D_g is the quadratic mean diameter; and e_{ij} denotes the error term (assumed $N(0, \sigma_e^2)$, mutually independent and independent of A_i).

In total, 11 potential predictor variables (listed in *Appendix* 1) were considered. For this purpose, unweighted linear least squares regression were used. The variables were tested according to four criteria:

1) Variables x_1 and x_2 are compulsory.
2) Subset models with a varying number of predictors, but with a similar (low) value of Mallow's C_p, are selected for further testing. If the model has p predictors, including the intercept, then $C_p = (k-p)(F-1)+p$, where k is the total number of (candidate) predictors and F is the F-statistic for testing the model in question against the model including all k predictors. Mallow's C_p combines estimates of bias and variance into a single measure of prediction error for the model.
3) The selected subset models are compared by a cross-validation method where each plot is systematically omitted in the parameter estimation (Andersen et al, 1982). A few predictor combinations with the lowest sum of squares of the prediction error of the excluded trees, are chosen as candidates for the final model.
4) The final choice is based on an evaluation of model behaviour, with due reference to the cross-validation results. For example, age may be selected as a final predictor according to the criteria 2 and 3, but, in combination with other predictor variables and due to the geographically biased origin of data, its inclusion may result in undesirable model behaviour.

The mixed-effect logarithmic stem volume model (eq [6]) accounts for within-plot correlation. The mean level, μ, the coefficients, b_k, and the inter-stand and single-tree variances, σ_A^2 and σ_e^2, respectively, are estimated by the restricted maximum likelihood method, using the Proc Mixed procedure of SAS.

By reverse transformation, the stem volume function can be expressed as

$$\hat{v}_{ij} = \exp\Big(\hat{\mu} + \sum_k \hat{b}_k x_{ijk} + \frac{s_A^2}{2} + \frac{s_e^2}{2}\Big) \qquad [7]$$

where symbols are as above, and the sample variance term $(s_A^2/2 + s_e^2/2)$ corrects the bias due to the logarithmic transformation. We refer to equation [7] as the general stem volume function.

The practical application of the stem volume function for a given stand, $\imath$, requires a prediction of the corresponding adjusted level, $A_\imath$. Based on trees sampled in stand $\imath$ for this purpose, the adjusted A-level can be calculated according to the following procedure:

Define

$$R_{\imath j} = Y_{\imath j} - \sum_k b_k x_{\imath jk} = A_\imath + e_{\imath j} \quad [8]$$

and suppose that the 'true' stem volumes of n sample trees in stand $\imath$ is known. Insertion of the estimates, $\hat{b}_k$, yields

$$\bar{R}_{\imath .} = \frac{1}{n} \sum_{j=1}^{n} \hat{R}_{\imath j} \quad [9]$$

and the estimator of $A_\imath$ with smallest mean square error is (cf Andersen, 1982)

$$\hat{A}_{\imath .} = w\hat{\mu} + (1 - w)\bar{R}_{\imath .} \quad [10]$$

where

$$w = \frac{\frac{1}{s_A^2}}{\frac{1}{s_A^2} + \frac{n}{s_e^2}} \quad [11]$$

is based on the sample variances, s_A^2 and s_e^2, for inter-stand and single-tree variation, respectively, $\hat{\mu}$ denotes estimated mean level, and n is number of trees sampled for level adjustment.

Using equation [10] to predict the stand level on the basis of n trees we obtain the single tree volume prediction with stand level adjustment

$$\hat{v}_{\imath j} = \exp\Big(\hat{A}_\imath + \sum_k \hat{b}_k x_{\imath jk} + \frac{w s_A^2}{2} + \frac{s_e^2}{2}\Big) \quad [12]$$

Compatible stem taper function

Combining the classic equations [2] and [3] yields the compatible stem taper model

$$d_l^2 = \frac{4v}{\pi h}(r+1)\left(1 - \frac{l}{h}\right)^r \quad [13]$$

where d_l designates the diameter at a given height l (above ground), h is the total tree height, and v is the stem volume.

For integers of the form exponent, r, equation [13] can be changed to a polynomial model of the variable (l/h). To allow for a varying form exponent, r, and thus improved flexibility, this model may be extended to the stem taper function (Madsen, 1985)

$$\hat{d}_l^2 = \begin{cases} \dfrac{4\hat{v}}{\pi h} \displaystyle\sum_{i=1}^{10} \hat{b}_i \left(\frac{l}{h}\right)^{i-4}, \\ 0.49\,\mathrm{m} \leq l \leq h, \\ \hat{d}_{0.49}^2 \\ \hat{h}_s \leq l < 0.49\,\mathrm{m} \end{cases} \quad [14]$$

where $\hat{d}_l$ denotes the predicted diameter l m above ground, $\hat{v}$ is the stem volume predicted by the stem volume function (eq [7]), $\hat{b}_i$ are coefficients ($i = 1, 2, \ldots, 10$), $\hat{d}_{0.49}$ denotes predicted diameter at 0.49 m above ground, and $\hat{h}_s$ is estimated stump height ($\hat{h}_s = 0.12 + d_{1.30}/4$).

Four restrictions are imposed on equation [14]:

1) the solid of revolution for the stem taper function (eq [14]) has to equal the volume of the stem volume function (eq [7]);
2) $\hat{d}_l = 0$ for $l = h$;
3) $\hat{d}_l = d_{1.30}$ for $l = 1.3$ m;
4) the derivative of $\hat{d}_l^2$ with respect to l has to equal zero for $l = h$.

Only restriction 1 is needed to provide compatible functions. Restrictions 2-4 give the stem taper function desirable properties. Restriction

4 ensures a smooth taper at the top. Compatibility may be achieved in two different ways: through taper calculations based on 1) stem volume estimates by the volume function with a general level μ (eq [7]), or based on 2) stem volume estimates adjusted according to stand conditions (eq [8-10]). Both types of compatible taper functions were developed.

Incorporating these four restrictions into equation [14] and correspondingly eliminating four b_i-coefficients (*Appendix* 2), ensures that the restrictions hold exactly for each tree. This results in the model

$$\frac{\pi h}{4\hat{v}} d_l^2 - a_{64} - a_{53}\left(\frac{l}{h}\right) - a_{42}\left(\frac{l}{h}\right)^2 - a_{31}\left(\frac{l}{h}\right)^3 = \sum_i b_i\left[\left(\frac{l}{h}\right)^{i-4} + a_{64+i} + a_{53+i}\left(\frac{l}{h}\right) + a_{42+i}\left(\frac{l}{h}\right)^2 + a_{31+i}\left(\frac{l}{h}\right)^3\right] + e_l$$

$$i = 1, 2, 3, 8, 9, 10 \qquad [15]$$

where the a-coefficients are summarized in *Appendix* 3. In generalized notation this becomes

$$y_l = \sum_i b_i x_{il} + e_l, \qquad [16]$$

$$i = 1, 2, 3, 8, 9, 10$$

The response variable y_l can be expressed as a function $\phi(d_l^2, \hat{v}, \hat{h}_s, h, l/h)$, and x_{il} as a function $\psi_i(\hat{h}_s, l/h)$.

The 'fixed' coefficients $b_1 - b_3$ and $b_8 - b_{10}$ are estimated for the material as a whole, using linear least squares. Thus, the 'fixed' coefficients remain constant across trees of the same species within the region. The number of 'fixed' coefficients may be reduced, if convenient, for the species under consideration (Madsen, 1985).

The 'variable' coefficients $b_4 - b_7$ are calculated seperately for each tree, in contrast to the 'fixed' coefficients, as

$$\hat{b}_{4+j} = a_{64-11j} + \sum_i \hat{b}_i\, a_{64-11j+i},$$

$$j = 0, 1, 2, 3,$$

$$i = 1, 2, 3, 8, 9, 10 \qquad [17]$$

(cf *Appendix* 2).

Then, based on $\hat{b}_1 - \hat{b}_{10}$, $\hat{v}$, h and $\hat{d}_{0.49}$, stem taper is determined by equation [14]. By integrating equation [14], an expression for merchantable stem volume with varying merchantable limit, a, can be deduced (Madsen, 1985) as

$$\tilde{v}_a = \hat{v}\left[\sum_i \hat{b}_i\left(\frac{1}{i-3}\left(\frac{l}{h}\right)^{i-3} + \left(\frac{0.49}{h}\right)^{i-3} \times \left(\frac{i-4}{i-3} - \frac{\hat{h}_s}{0.49}\right)\right) + \hat{b}_3\left(\ln\left(\frac{l}{0.49}\right) + 1 - \frac{\hat{h}_s}{0.49}\right)\right],$$

$$i = 1, 2, 4, 5, \ldots, 10 \qquad [18]$$

where $\tilde{v}_a$ is predicted merchantable stem volume, l denotes the height above ground where the merchantable limit a occurs, and other symbols are as previously described. The height l corresponding to a given merchantable limit may be calculated using an iterative procedure (Madsen, 1985).

RESULTS

Stem volume function

For the stem volume function the selection procedure indicated 4-5 variables to be the optimal number; no improvement of C_p or the cross-validation results occurred by including more variables. In addition to the two compulsory predictors, all relevant combinations of x-variables included x_4 (= $\ln(h/[h - 1.30])$), whereas stand height, H_g, only appeared in inferior subset

Table II. Coefficients in the stem volume function for oak.

Range	$0.008 < d_{1.30} < 1.114$ m,		$2.0 < h < 32.4$ m,		$0.024 < D_g < 0.875$ m		
Coeff	μ	b_1	b_2	b_4	b_{25}	σ_A^2	σ_e^2
Est	−1.876293	1.876423	1.219715	1.306483	0.0968591	0.004919	0.000598

Table III. 'Fixed' coefficients in the stem taper function when the function is based on general level (GL) or adjusted level (AL) stem volumes for oak.

Range	$0.035 < d_{1.30} < 1.114$ m, $5.4 < h < 32.4$ m, $0.0034 < v < 11.1704$ m^3, $0.056 < D_g < 0.875$ m, $0.0032 < \hat{v}_{AL} < 13.8073$ m^3, $0.0033 < \hat{v}_{GL} < 14.1738$ m^3					
Coeff	b_1	b_2	b_3	b_8	b_9	b_{10}
AL	2.947623E-5	−2.846482E-3	1.115932E-1	−46.54280	46.76369	−16.04021
GL	2.412516E-5	−2.192686E-3	8.259829E-2	81.07845	−47.21749	11.10291

models. To account for temporal changes in stand conditions, a fourth and/or a fifth variable, therefore, had to include stand diameter, D_g, or stand age, T. For oak, subset models including T performed well. This may be attributed to the uneven geographical distribution of plots. For most other species, T seems to be a predictor of dubious quality (Madsen, 1987), except for Norway spruce (Madsen and Heusèrr, 1993). Based on these results, terms with T were excluded.

The final model includes x_1, x_2, x_4 and x_{25} as predictor variables (table II). Note that the stem volume function, provided constant h and D_g, has the desirable property that the stem form factor decreases with increasing diameter at breast height (ie, $\hat{b}_1 < 2$).

Stem taper function

Preliminary calculations revealed that one particular plot (sample plot RA, Esrum) introduces a severe bias in the stem taper function. This may be attributed to two factors: 1) trees from this plot are by far the smallest (the plot is represented by only one measurement occasion; $D_g = 2.4$ cm, $H_g = 3.6$ m), and 2) rounding diameter measurements to the nearest even millimetre reduces precision compared to the rest of the material. When this plot is omitted the results improve considerably, and it is thus left out in the estimation procedure for the stem taper function.

Results are given in table III, both for the general level function and for the adjusted level function. Owing to restriction 1 for equation [14] the stem taper function has the same desirable property as the stem volume function that, provided constant h and D_g, the stem form factor decreases with increasing diameter at breast height.

DISCUSSION OF SPECIFIC RESULTS

Stem volume function

Model assumptions for the stem volume function include parallel response planes whose position is determined by stand-specific levels, ie, levels that are specific to site as well as to thinning treatment. An F-test revealed a significant, but relatively small interaction between $\ln(d_{1.30})$ and stand level (A in eq [6]). Considering the need for a function easy to use in practice, this irregularity was ignored. Unfortunately, the material leaves no real opportunities to test a possible (expected) pure effect of thinning.

A closer examination of plots with an almost equal proportion of forked and unforked trees revealed that the stem volume of forked trees is underpredicted compared to unforked trees. A level adjustment for forking was sufficient to account for the differences, but inconvenient to handle in practice and therefore not included.

According to the variance components, the coefficient of variation (CV) for trees within a stand is 7.0% ($s_e^2 = 0.004919$), and the CV for stand means is 2.4% ($s_A^2 = 0.000598$). For prediction of single tree volume, the CV is, however, somewhat larger than 7.0% because of the uncertainty in the estimate of the stand level. Using the stand level adjustment (eq [12]) when predicting single tree volumes, the relative prediction error is

$$rpe_{\text{single}} = \sqrt{\left(\frac{1}{s_A^2} + \frac{n}{s_e^2}\right)^{-1} + s_e^2} \qquad [19]$$

where n is the number of trees sampled for level adjustment.

The relative prediction error for the sum of the total stem volumes of $a = 1, 2, \dots, m$ trees may similarly be shown to be

$$rpe = \sqrt{\frac{1}{\frac{1}{s_A^2} + \frac{n}{s_e^2}} + \frac{\sum_{a=1}^{m} \hat{v}_{ia}^2}{\left(\sum_{a=1}^{m} \hat{v}_{ia}\right)^2} s_e^2} \qquad [20]$$

$$= \sqrt{\frac{1}{\frac{1}{s_A^2} + \frac{n}{s_e^2}} + \frac{1}{m}\left[1 + \left(\frac{s_{\hat{v},i}}{\bar{v}_i}\right)^2\right] s_e^2} \qquad [21]$$

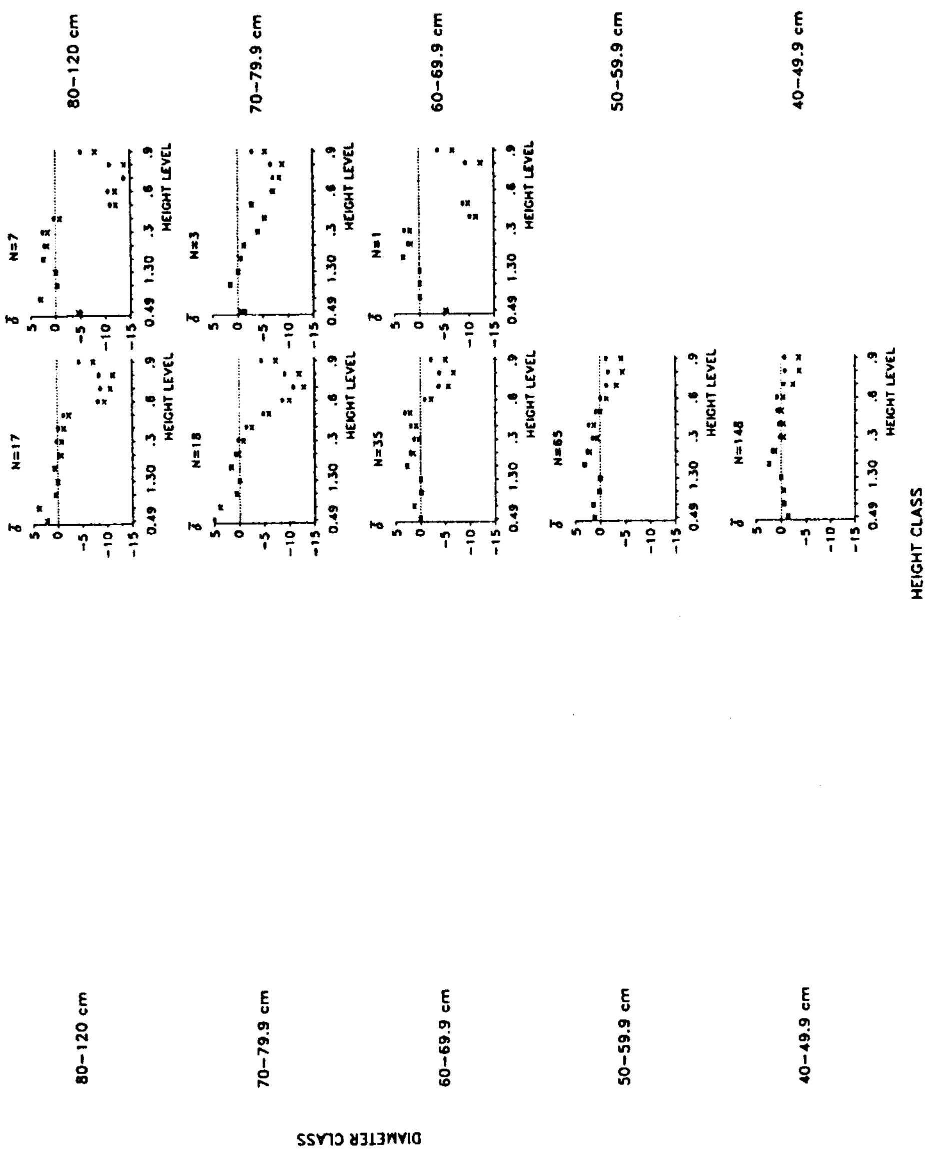
where $\hat{v}_{ia}$ denotes the predicted stem volume of the individual m trees. When all m trees are of equal size, rpe assumes a minimum value for any given number, n, of sample trees. When one tree is much larger than the remaining $m - 1$ very small trees, rpe converges towards its maximum value ($\approx rpe_{\text{single}}$) for any given number of sample trees.

The coefficient of variation for predicted stem volumes ($\text{CV}(\hat{v}_i) = s_{\hat{v},i}/\bar{v}_i$ in eq [21]) increases with increasing age and decreases with increasing thinning grade. For oak on sites such as those included in this study, $\text{CV}(\hat{v}_i)$ typically ranges from 10% in heavily thinned stands to 20% in lightly thinned stands at the age of 75 years. In heavily thinned stands $\text{CV}(\hat{v}_i)$ reaches 40% at the age of 150 years. As an example, let $\text{CV}(\hat{v}_i) < 50\%$, $50 < m < 2500$, and $5 < n < 25$. Then rpe_{single} varies between 7.1 and 7.3%, and rpe between 1.2 and 2.2%.

Stem taper function

In addition to the properties implied by model restrictions 1-4, the stem taper function should generally predict the diminishing diameter from the ground to the top of the tree, and should not predict negative values of d_l^2.

These properties were examined by predictions of d_l^2 for all ($d_{1.30}$, h, D_g)-combinations occurring in the material; d_l^2 was predicted for all heights, l, at which


HEIGHT CLASS
DIAMETER CLASS
80–120 cm
70–79.9 cm
60–69.9 cm
50–59.9 cm
40–49.9 cm
N=17
N=7
N=18
N=3
N=35
N=1
N=65
N=148
HEIGHT LEVEL
0.49 1.30 .3 .6 .9

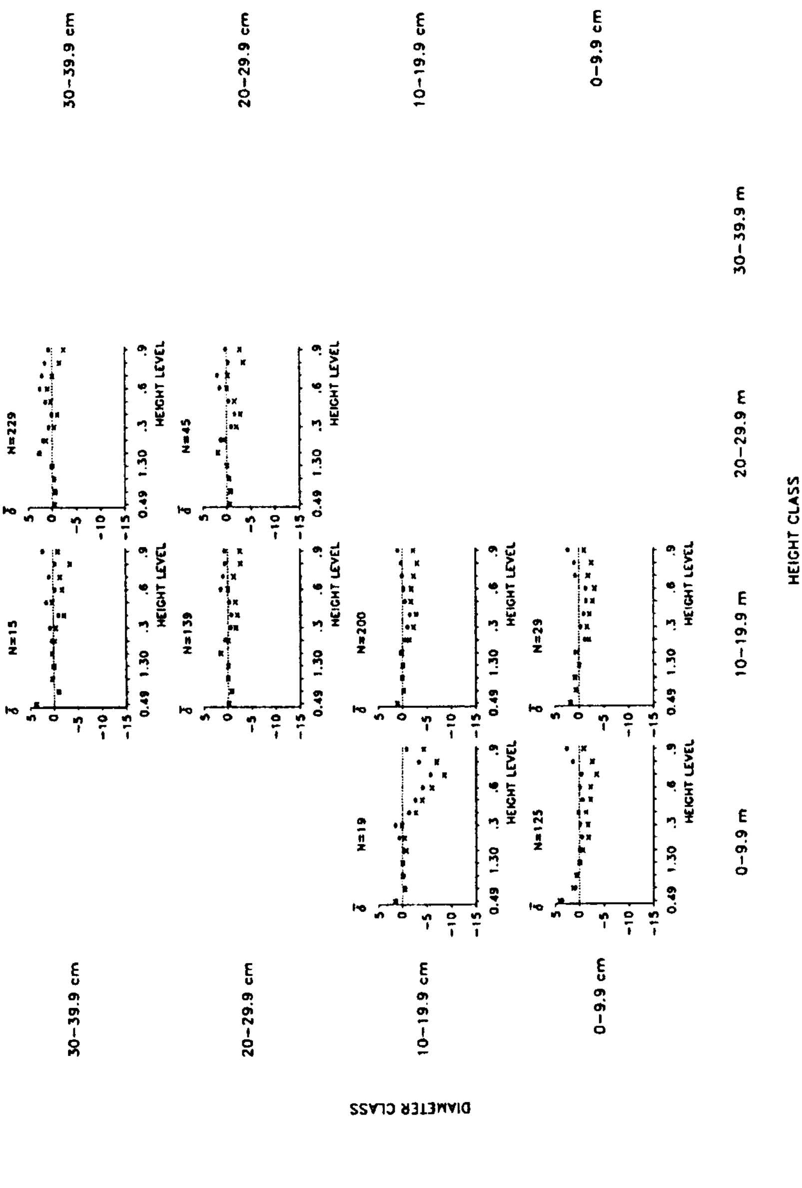


Fig 3. Mean relative deviations, $\bar{\delta}$, of stem taper function from the original mesurements. Legend: stem taper based on: ● adjusted level stem volume; × general level stem volumes; N indicates number of observations in each size class. Further explanations are given in text and in legend of table IV. Five observations are outside axes ranges, and these are shown in table V.

Table IV. Evaluation of stem taper function based on adjusted level (AL) and general level (GL) stem volumes. Predictions are compared to original measurements for each measurement height, l. Legend: $\bar{\delta}$ is mean value of individual relative deviations, where $\delta = 100(d_l - \hat{d}_l)/d$ and $d = d_{1.30}$; $\bar{d}$ is mean value of breast height diameters.

Rel diam	*All* 1095 *trees*				882 *unforked trees*				213 *forked trees*			
	AL		*GL*		*AL*		*GL*		*AL*		*GL*	
	$\bar{\delta}$	$\frac{s_\delta}{\bar{d}}$	$\bar{\delta}$	$\frac{s_\delta}{\bar{d}}$	$\bar{\delta}$	$\frac{s_\delta}{\bar{d}}$	$\bar{\delta}$	$\frac{s_\delta}{\bar{d}}$	$\bar{\delta}$	$\frac{s_\delta}{\bar{d}}$	$\bar{\delta}$	$\frac{s_\delta}{\bar{d}}$
$d_{0.9}/d$	0.4	14	−2.8	14	−0.1	15	−3.3	14	2.3	10	−0.7	10
$d_{0.8}/d$	−0.2	21	−3.3	21	−1.2	22	−4.4	21	4.3	14	1.3	14
$d_{0.7}/d$	−0.1	24	−2.4	24	−1.5	25	−3.8	25	5.4	14	3.3	14
$d_{0.6}/d$	0.3	24	−1.2	24	−1.0	25	−2.6	25	5.9	15	4.5	15
$d_{0.5}/d$	0.1	21	−1.1	21	−0.9	21	−2.1	21	4.2	15	3.1	15
$d_{0.4}/d$	−0.2	17	−1.4	17	−0.7	18	−2.0	18	1.9	13	0.8	13
$d_{0.3}/d$	0.1	14	−1.1	14	−0.2	15	−1.3	15	1.1	10	0.0	10
$d_{0.2}/d$	0.8	12	0.1	13	0.6	14	−0.2	14	1.9	8	1.3	8
$d_{0.1}/d$	1.6	12	1.4	12	1.4	13	1.2	13	2.6	7	2.5	7
$d_{1.14}/d$	−0.1	9	0.0	9	0.0	10	0.1	10	−0.3	4	−0.3	4
$d_{0.81}/d$	0.0	11	0.1	11	0.0	12	0.1	12	−0.1	8	−0.1	8
$d_{0.49}/d$	0.6	19	0.6	19	0.8	21	0.8	21	−0.3	12	−0.4	12

diameters were measured. The computations showed no cases of an increasing diameter by increasing height above ground, after exclusion of the sample plot RA. In contrast, for 426 trees, measured diameters increased with height in one or more parts of the tree. For 272 of these trees, the increase in diameter was either very small ($\leq$ 1 mm) or due to forking (the procedure for calculating a joint sectional area will often cause increasing diameters around the point of forking). Different types of irregularities, such as wounds, coarse bark areas and epicormic branches, may be responsible for the rest. No negative predictions of d_l^2 were observed.

The performance of the stem taper function has been evaluated in terms of relative deviations, δ, from the original measurements, where $\delta = 100(d_l - \hat{d}_l)/d_{1.30}$, for each measurement height l. Mean values appear in table IV and figure 3.

The stem taper function generally predicts well for the lower and economically important part of the stem. For the material as a whole (table IV), mean deviations are positive (ie, predicted diameter too small) in the lower one third of the stem, and negative above. For forked trees, however, mean deviations are generally positive throughout, consistent with stem volume (under) predictions. The deviations have minimum standard deviations near breast height and near the tree top, corresponding to the restrictions forcing the stem ta-

Table V. Five observations outside axes range in figure 3.

Size class	*Height level*	$\bar{\delta}_{AL}$	$\bar{\delta}_{GL}$
$20 - 29.9$ m $\times$ $70 - 79.9$ cm	0.49	5.0	5.1
$30 - 39.9$ m $\times$ $60 - 69.9$ cm	0.6	−16.3	−17.6
$30 - 39.9$ m $\times$ $60 - 69.9$ cm	0.7	−19.3	−21.3
$30 - 39.9$ m $\times$ $70 - 79.9$ cm	0.81	6.9	6.7
$30 - 39.9$ m $\times$ $80 - 120$ cm	0.7	−13.7	−15.7

per through $d_{1.30}$ at breast height and zero at the tree top.

For large trees ($d_{1.30} > 60$ cm, $h > 20$ m), the performance of the stem taper function deteriorates, in particular for the upper and economically unimportant part of the stem (fig 3). The adjustment of volume level generally results in decreased taper deviations. The stem taper deviations show similar patterns for the general level and adjusted level models. This indicates that the pattern is a charateristic of the model, rather than a consequence of which type of stem volume estimates (adjusted or general level) the stem taper function is based on.

GENERAL DISCUSSION

The general discussion focuses on four issues: *Material, Methods, Results in relation to previous work* and *Application in practice.*

Material

The material covers a considerable range of individual tree sizes and stand conditions, indicating good representation. However, the data, and thus the volume and stem taper functions, are subject to four limitations.

First, the uneven geographical distribution of plots implies that the functions have not been verified for trees growing under poor conditions, ie, notably plantations on sandy soils in Jutland. Second, only thinned trees were sampled, and by subjective procedures. In consequence, the functions may not represent well the extremes in young stands and large crop trees in old stands. Furthermore, temporal and personal differences in sampling procedures may have affected the inter-stand variation. Third, the trees were sampled over a long period of time, and may represent other than current thinning practices. Fourth, the use of the relative method limits procedures for stem volume calculations. This introduces a bias in the volume function because the 'true' stem volume (eq [1]) as calculated in this paper, results in an overestimation of approximately 0.4% (Madsen, 1985, 1987). On the other hand, $d_{1.30}$ and h are easy to measure in practice.

Methods

A problem with an analysis of this kind is that trees within a stand are not statistically independent. This is reflected by the choice of cross-validation method in the selection procedure for predictor variables. An entire stand was omitted in each step, thus leading to fairly realistic prediction errors on which the selection is based. For practical reasons an initial selection was, however, based on Mallow's C_p which, by its construction, behaves much like a cross-validation with single tree omissions. This method favours models with too many stand variables, but was considered sufficiently reliable for a preliminary selection.

Another problem is that diameter measurements along a single stem may be correlated. This is indirectly accounted for by the use of smooth functions in the stem taper model (eq [14]), and by the use of tree volume as a covariate. Alternatively, within-tree correlations may be modelled directly (see for example Gregoire and Schabenberger, 1996), in which case restrictions 3 and 4 should be abandoned.

Results in relation to previous work

The accuracy of predictions from the stem volume function for oak is comparable to similar functions for three broadleaved species (Madsen, 1987) and six conifers (Madsen, 1987; Madsen and Heusèrr, 1993). The lack of parallel response planes and a potential systematic effect of forked trees, also agree with previous results.

Comparable stem taper functions for other broadleaved species are not available, but compared to conifers (Madsen, 1985; Madsen and Heusèrr, 1993) the stem taper function for oak performs relatively well. For conifers, mean deviations are generally below 1%, the few forks being excluded. Mean deviations ranging up to 3.3% for oak, where the number of forked trees were substantial, is therefore a satisfactory result.

Application in practice

Being based on stem form considerations as well as easily measurable tree variables, both the stem volume and the stem taper models are accessible and easy to use in practice. Worked examples to help users check their implementation of the functions are given in *Appendix 4* (table VI).

A consequence of the model concept is that bias in the stem volume function will be transferred directly to the stem taper function. By choice of adequate volume function, this weakness is of minor importance and is for practical purposes outweighed by the advantages of compatibility. For a more detailed discussion of the model concept, see Madsen (1985, 1987).

For applications in science and in practice, it is a particularly important feature that the stand specific level A_i is operational. This considerably increases the applicability of the functions for use in, for example, growth modelling and forest management. For forest management, the information from the stem taper function on log mid-diameters is very valuable.

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APPENDIX 1

The x-variables considered for the stem volume function are:

$$x_1 = \ln(d_{1.30});$$
$$x_2 = \ln(h);$$
$$x_3 = \ln(D_g);$$
$$x_4 = \ln\left(\frac{h}{h-1.30}\right);$$
$$x_5 = \ln^2(D_g);$$
$$x_6 = \ln(H_g);$$
$$x_7 = \ln^2(H_g);$$
$$x_8 = \ln(T);$$
$$x_9 = \ln^2(T);$$
$$x_{24} = D_g;$$
$$x_{25} = D_g^2.$$

The x-variables x_{10}-x_{23} (Madsen, 1987) are for merchantable volume functions and volume functions including diameter at 3 or 6 m above ground. Therefore, they are not taken into consideration in this paper.

APPENDIX 2

Incorporating the four restrictions imposed on the stem taper function (cf Madsen, 1985). Symbols are as explained in the main text.

Given the stem taper function (equation [14] in the main text)

$$\hat{d}_l^2 = \begin{cases} \dfrac{4\hat{v}}{\pi h}\displaystyle\sum_{i=1}^{10}\hat{b}_i\left(\frac{l}{h}\right)^{i-4}, \\ 0.49\,\mathrm{m} \le l \le h \\ \\ \hat{d}_{0.49}^2, \\ \hat{h}_s \le l < 0.49\,\mathrm{m} \end{cases} \quad \text{[A1]}$$

the total stem volume excluding stump (ie, the stem volume between heights $\hat{h}_s$ and h) is

$$\begin{aligned}\tilde{v} &= \int_{\hat{h}_s}^{h}\frac{\pi}{4}\hat{d}_l^2\,\mathrm{d}l \\ &= \frac{\hat{v}}{h}\int_{\hat{h}_s}^{0.49}\sum_{i=1}^{10} b_i\left(\frac{0.49}{h}\right)^{i-4}\mathrm{d}l \qquad \text{[A2a]} \\ &\quad + \frac{\hat{v}}{h}\int_{0.49}^{h}\sum_{i=1}^{10} b_i\left(\frac{l}{h}\right)^{i-4}\mathrm{d}l \\ &= \hat{v}\left[\left(\sum_i b_i\left(\frac{1}{i-3} + \left(\frac{i-4}{i-3} - \frac{\hat{h}_s}{0.49}\right)\right.\right.\right. \\ &\quad \left.\left.\times\left(\frac{0.49}{h}\right)^{i-3}\right)\right) \qquad \text{[A2b]} \\ &\quad \left. + b_3\left(1 - \ln\left(\frac{0.49}{h}\right) - \frac{\hat{h}_s}{0.49}\right)\right]\end{aligned}$$

where $i = 1, 2, 4, 5, 6, 7, 8, 9, 10$ in equation [A2b]. Defining a_1-a_{10} as

$$a_i = \begin{cases} \frac{1}{i-3} + \left(\frac{0.49}{h}\right)^{i-3}\left(\frac{i-4}{i-3} - \frac{\hat{h}_s}{0.49}\right), \\ i = 1,2,4,5,6,7,8,9,10 \\ 1 - \ln\left(\frac{0.49}{h}\right) - \frac{\hat{h}_s}{0.49}, \\ i = 3 \end{cases} \quad [A3]$$

the four restrictions may be expressed as follows:

1) the compatibility restriction:

$$\sum_{i=1}^{10} b_i a_i = 1 \quad [A4]$$

2) the restriction on top diameter:

$$\sum_{i=1}^{10} b_i = 0 \quad [A5]$$

3) the restriction on breast height diameter:

$$\sum_{i=1}^{10} b_i \left(\frac{1.3}{h}\right)^{i-4} = \frac{\pi h}{4\hat{v}} d_{1.30}^2 \quad [A6]$$

4) the derivative restriction:

$$\sum_{i=1}^{10} b_i (i-4) = 0 \quad [A7]$$

Using equations [A4-A7], four b-coefficients may be eliminated in the stem taper function (eq [A1]). This will ensure that the four restrictions hold exactly for each tree.

Using equation [A5] to eliminate b_4 in equations [A4] and [A6] yields

$$\sum_i b_i (a_i - a_4) = 1 \quad [A8]$$

$$\sum_i b_i \left(\left(\frac{1.3}{h}\right)^{i-4} - 1\right) = \frac{\pi h}{4\hat{v}} d_{1.30}^2 \quad [A9]$$

$$\sum_i b_i (i-4) = 0 \quad [A10]$$

where $i = 1,2,3,5,6,7,8,9,10$.

Using equation [A10] to eliminate b_5 in equations [A8] and [A9] yields

$$\sum_i b_i \big((a_i - a_4) - (i-4)(a_5 - a_4)\big) = 1 \quad [A11]$$

and

$$\sum_i b_i \left(\left(\frac{1.3}{h}\right)^{i-4} - 1 - (i-4)\left(\frac{1.3}{h} - 1\right)\right) = \frac{\pi h}{4\hat{v}} d_{1.30}^2 \quad [A12]$$

respectively, where $i = 1,2,3,6,7,8,9,10$.

To eliminate b_6, define

$$a_{10+i} = (a_i - a_4) - (i-4)(a_5 - a_4), \quad i = 1,2,3,6,7,8,9,10 \quad [A13]$$

and

$$a_{20+i} = \left(\frac{1.3}{h}\right)^{i-4} - 1 - (i-4)\left(\frac{1.3}{h} - 1\right), \quad i = 1,2,3,6,7,8,9,10 \quad [A14]$$

Then equation [A11] may be expressed as

$$\sum_i b_i a_{10+i} = 1 \quad [A15]$$

and ([A12]) as

$$\sum_i b_i a_{20+i} = \frac{\pi h}{4\hat{v}} d_{1.30}^2 \qquad \text{[A16]}$$

where $i = 1, 2, 3, 6, 7, 8, 9, 10$. Using [A15] to eliminate b_6 in equation [A16] yields

$$\sum_i b_i \left(a_{20+i} \frac{a_{16}}{a_{26}} - a_{10+i} \right) = \frac{\pi h}{4\hat{v}} d_{1.30}^2 \frac{a_{16}}{a_{26}} - 1 \qquad \text{[A17]}$$

where $i = 1, 2, 3, 7, 8, 9, 10$.

To eliminate b_7, define

$$a_{31} = \frac{\frac{\pi h}{4\hat{v}} d_{1.30}^2 \frac{a_{16}}{a_{26}} - 1}{a_{27} \frac{a_{16}}{a_{26}} - a_{17}} \qquad \text{[A18]}$$

and

$$a_{31+i} = -\frac{a_{20+i} \frac{a_{16}}{a_{26}} - a_{10+i}}{a_{27} \frac{a_{16}}{a_{26}} - a_{17}}, \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A19]}$$

Then [A17] may be expressed as

$$b_7 = a_{31} + \sum_i b_i a_{31+i}, \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A20]}$$

Next, define

$$a_{42} = (1 - a_{17} a_{31}) / a_{16} \qquad \text{[A21]}$$

and

$$a_{42+i} = -(a_{10+i} + a_{31+i} a_{17}) / a_{16}, \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A22]}$$

Inserting equation [A20] into equation [A15] yields

$$b_6 = a_{42} + \sum_i b_i a_{42+i}, \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A23]}$$

To eliminate b_5, define

$$a_{53} = -(2a_{42} + 3a_{31}) \qquad \text{[A24]}$$

and

$$a_{53+i} = -(i - 4 + 2a_{42+i} + 3a_{31+i}), \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A25]}$$

Inserting equations [A20] and [A23] into equation [A10] yields

$$b_5 = a_{53} + \sum_i b_i a_{53+i}, \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A26]}$$

To eliminate b_4, define

$$a_{64} = -(a_{53} + a_{42} + a_{31}) \qquad \text{[A27]}$$

and

$$a_{64+i} = -(1 + a_{53+i} + a_{42+i} + a_{31+i}), \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A28]}$$

Inserting equations [A20], [A23] and [A26] into equation [A5] yields

$$b_4 = a_{64} + \sum_i b_i a_{64+i}, \quad i = 1, 2, 3, 8, 9, 10 \qquad \text{[A29]}$$

Now, the a-coefficients are defined and the four restrictions imposed on the stem taper function are expressed through equations [A20], [A23], [A26] and [A29]. The restrictions are now expressed in the form of b-coefficients ready to be eliminated in the stem taper function (eq [A1]).

Inserting equations [A20], [A23], [A26] and [A29] into equation [A1] yields

$$\frac{\pi h}{4\hat{v}} \hat{d}_l^2 - a_{64} - a_{53}\left(\frac{l}{h}\right) - a_{42}\left(\frac{l}{h}\right)^2 - a_{31}\left(\frac{l}{h}\right)^3 = \sum_i b_i \left[\left(\frac{l}{h}\right)^{i-4} + a_{64+i} + a_{53+i}\left(\frac{l}{h}\right) + a_{42+i}\left(\frac{l}{h}\right)^2 + a_{31+i}\left(\frac{l}{h}\right)^3\right],$$
$$i = 1, 2, 3, 8, 9, 10 \qquad \text{[A30]}$$

For the least squares estimation procedure equation [A30] leads to equation [15] in the main text.

APPENDIX 3

The a-coefficients of the stem taper function are:

$$a_i = \frac{1}{i-3} + \left(\frac{0.49}{h}\right)^{i-3}\left(\frac{i-4}{i-3} - \frac{\hat{h}_s}{0.49}\right), \quad i = 1, 2, 4, 5, 6, 7, 8, 9, 10;$$

$$a_3 = 1 - \ln\left(\frac{0.49}{h}\right) - \frac{\hat{h}_s}{0.49};$$

$$a_{10+i} = (a_i - a_4) - (i-4)(a_5 - a_4), \quad i = 1, 2, 3, 6, 7, 8, 9, 10;$$

$$a_{20+i} = \left(\frac{1.3}{h}\right)^{i-4} - 1 - (i-4)\left(\frac{1.3}{h} - 1\right), \quad i = 1, 2, 3, 6, 7, 8, 9, 10;$$

$$a_{31} = \frac{\frac{\pi h}{4\hat{v}} d_{1.30}^2 \frac{a_{16}}{a_{26}} - 1}{a_{27}\frac{a_{16}}{a_{26}} - a_{17}};$$

$$a_{31+i} = -\frac{a_{20+i}\frac{a_{16}}{a_{26}} - a_{10+i}}{a_{27}\frac{a_{16}}{a_{26}} - a_{17}}, \quad i = 1, 2, 3, 8, 9, 10;$$

$$a_{42} = \frac{1 - a_{17}a_{31}}{a_{16}};$$

$$a_{42+i} = -\frac{a_{10+i} + a_{17}a_{31+i}}{a_{16}}, \quad i = 1, 2, 3, 8, 9, 10;$$

$$a_{53} = -(2a_{42} + 3a_{31});$$

$$a_{53+i} = -(i - 4 + 2a_{42+i} + 3a_{31+i}), \quad i = 1, 2, 3, 8, 9, 10;$$

$$a_{64} = -(a_{53} + a_{42} + a_{31});$$

$$a_{64+i} = -(1 + a_{53+i} + a_{42+i} + a_{31+i}), \quad i = 1, 2, 3, 8, 9, 10.$$