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Original article

Influence of cross section dimensions on Timoshenko's shear factor – Application to wooden beams in free-free flexural vibration

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Abstract – This study was designed to examine the influence of the cross section height-to-width ratio on Timoshenko's shear factor. This factor was introduced to account for the irregular shear stress and shear strain distribution over the cross section. A new theoretical formulation of the shear factor is thus proposed to assess rectangular cross sections of orthotropic material such as wood. Numerical simulations were performed to examine shear factor variations with respect to the height-to-width ratio. The influence of the cross section size on the first five flexural vibration frequencies is also discussed.

Timoshenko's shear factor / flexural vibration / wood

Résumé – Influence des dimensions de la section transverse sur le facteur de cisaillement de Timoshenko. Applications aux poutres en bois en vibration de flexion libre–libre. L'objectif de cette étude est d'examiner l'influence du ratio hauteur sur épaisseur de la section droite sur le facteur de cisaillement de Timoshenko. Ce facteur est utilisé afin de prendre en compte le fait que les contraintes et les déformations de cisaillement ne sont pas uniformément réparties dans la section droite. Une nouvelle formulation théorique du facteur de cisaillement est alors proposée dans le cas d'une section droite rectangulaire pour un matériau orthotrope comme le bois. Des simulations numériques sont réalisées de manière à examiner la variation du facteur de cisaillement en fonction du ratio hauteur sur épaisseur. L'influence de l'effet de dimension de la section droite sur les cinq premières fréquences de vibration de flexion est également discutée.

facteur de cisaillement de Timoshenko / vibration de flexion / bois

1. INTRODUCTION

Free-free flexural vibration tests can be used to accurately determine the elastic constants of wooden beams such as Young's modulus and the shear modulus [4]. The shear effect and rotary inertia can be taken into account to extend the range of applicability of the Euler-Bernoulli theory of beams [7]. These effects are incorporated in Timoshenko's beam equation. This formula has been the focus of considerable attention in the literature, but few studies have investigated the validity range of Timoshenko's beam equation when the height/width ratio of the cross section varies between that of a thin vibrating beam to that of a thick vibrating plate. Cowper [2] derived Timoshenko's beam equation by integration of equations based on the three-dimensional elasticity theory. A new shear factor formula was proposed which integrates Poisson's ratio for an isotropic material but the cross section size effect was not taken into consideration [2]. In this paper, a theoretical formulation for the shear factor is proposed for orthotropic materials such as wood with a rectangular cross section. This new theoretical formulation includes relations based on the three-dimensional elasticity theory and takes the size effect of the cross section into account. Numerical shear factor values are calculated according to variations in the height/width ratio of the cross section. The influence of the cross section size effect on the first five vibration frequencies is also discussed.

2. THEORETICAL FORMULATION

Let us consider an orthotropic and homogeneous beam in free-free bending vibration. The governing equation of motion, as formulated by Timoshenko [7] for an isotropic material, is as follows:

$$E_X I_{Gz}\frac{\partial^4 \nu}{\partial x^4} - \rho I_{Gz}\left(1 + \frac{E_X}{KG_{XY}}\right)\frac{\partial^4 \nu}{\partial x^2 \partial t^2} + \frac{\rho^2 I_{Gz}}{KG_{XY}}\frac{\partial^4 \nu}{\partial t^4} + \rho S\frac{\partial^4 \nu}{\partial t^2} = 0 \quad (1)$$

where E_X is the longitudinal modulus of elasticity, I_{Gz} the cross section inertia, ρ the density, K the Timoshenko shear factor, G_{XY} the shear modulus, S the cross section area and ν the particle motion on the axis (OY). Equation (1) takes the effects of shear deflection and rotary inertia into account. Coefficient K, which is a dimensionless quantity that is dependent on the shape of the cross section, is introduced to account for the irregular shear stress and shear strain distribution over the cross

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section [2]. Timoshenko [8] defined K as the ratio of the average shear strain on a section to the shear strain at the centroid. The K value is thus 2/3 for a beam with a rectangular cross section [8]. However, several authors have proposed other K value estimates [2–4, 10] and the consensus value seems to be 5/6 [11].

The K value for a rectangular cross section can be calculated considering the elementary work dw associated with the shear stress σ_{xy} acting on a beam element of length dx [5]:

$$\mathrm{d}w = \frac{1}{2G_{xy}}\left[\iint_S \sigma_{xy}^2 \mathrm{d}S\right]\mathrm{d}x\,. \quad (2)$$

Timoshenko [9], Lekhnitskii [6] and Laroze [5] developed theoretical formulations on the basis of a three-dimensional state of stress. The shear stress distributions σ_{xy} and σ_{xz} are derived from the function $\varphi(y,z)$. In particular, it is shown that σ_{xy} could be written as (3) for a rectangular cross section [9]:

$$\sigma_{xy} = \frac{T_y}{2I_{Gz}}\left(\frac{h^2}{4}-y^2\right)+\frac{\partial\varphi(y,z)}{\partial z}\,. \quad (3)$$

This function is commonly expressed in the form of a double Fourier series. The expression of the function $\varphi(y,z)$ is given by Timoshenko [9] and reformulated for an orthotropic material:

$$\varphi(y,z) = -2\frac{\nu_{xz}G_{xy}}{E_x}\frac{T_y}{I_{Gz}}\frac{e^3}{\pi^4}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n-1}\cos\dfrac{(2m+1)\pi y}{h}\sin\dfrac{2n\pi z}{e}}{(2m+1)n\left[(2m+1)^2\dfrac{e^2}{4h^2}+n^2\right]} \quad (4)$$

where e is the width of the cross section. This last expression is mathematically correct but not easy to practically apply. A simplified formula, based on an analogy with membrane material, could be proposed when the height and width of the cross section are in the same scale [9]:

$$\varphi(y,z) = -\frac{\nu_{xz}G_{xy}}{E_x}\frac{T_y}{I_{Gz}}\left(\frac{h^2}{4}-y^2\right)\left(\frac{e^2}{4}-z^2\right)\left(\frac{P_1\left(\frac{h}{e}\right)}{e^2P_2\left(\frac{h}{e}\right)}z+\frac{4}{e^4P_2\left(\frac{h}{e}\right)}z^3\right) \quad (5)$$

where P_1 and P_2 are two polynomials of the form:

$$P_1\left(\frac{h}{e}\right) = \frac{1}{11}+8\left(\frac{h}{e}\right)^2 \text{ and}$$

$$P_2\left(\frac{h}{e}\right) = \left(\frac{1}{7}+\frac{3}{5}\left(\frac{h}{e}\right)^2\right)P_1\left(\frac{h}{e}\right)+\frac{1}{21}+\frac{9}{35}\left(\frac{h}{e}\right)^2\,. \quad (6)$$

By deriving equation (5) and applying equation (3), the shear stress σ_{xy} is thus expressed as:

$$\sigma_{xy} = \frac{T_y}{2I_{Gz}}\left(\frac{h^2}{4}-y^2\right) \times\left(1-2\frac{\nu_{xz}G_{xy}}{E_x}\left(\frac{P_1\left(\frac{h}{e}\right)}{4P_2\left(\frac{h}{e}\right)}+\frac{3\left(1-P_1\left(\frac{h}{e}\right)\right)}{e^2P_2\left(\frac{h}{e}\right)}z^2-\frac{20}{e^4P_2\left(\frac{h}{e}\right)}z^4\right)\right). \quad (7)$$

Formula (7) is used with equation (2) to obtain analytic expressions of shear factor K^* for an orthotropic and homogeneous material when the dimensions of the cross section are in the same scale:

$$K^* = \frac{5}{6}\frac{1}{1+\left(\frac{\nu_{xz}G_{xy}}{E_x}\right)^2\dfrac{23+9P_1\left(\frac{h}{e}\right)\left(6+7P_1\left(\frac{h}{e}\right)\right)}{315P_2\left(\frac{h}{e}\right)^2}} \quad (8)$$

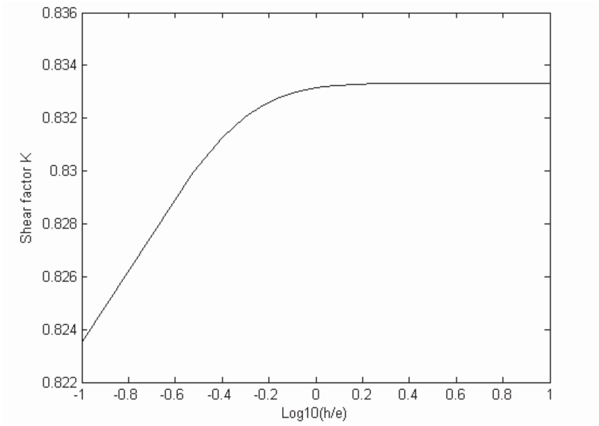

Figure 1. Variations in the shear factor K^* according to the base 10 logarithm of the h/e ratio.

The difference between the shear factor K^* calculated from the exact expression (4) and from the simplified formula (5) is less than 2% when the ratio $h/e \geq 2$ and it can be neglected when h/e is reaching 10. However, the difference is 6% when h/e is reaching 1/10.

3. NUMERICAL TRENDS FOR WOOD MATERIAL

Equation (8), as formulated above, can reveal trends in the shear factor K^* with variations in cross section dimensions. We used common wood mechanical characteristic values to highlight the shear factor patterns. The longitudinal modulus of elasticity E_X values were thus 14 000 MPa, the shear modulus G_{XY} values in the LT plane were 900 MPa and the Poisson's ratio ν_{XZ} values in the LR plane were 0.39. The K^* values were plotted when the height h to width e ratio of the cross section ranged from 0.1 to 10 (Fig. 1). This variation range was fixed in compliance with Timoshenko's approximation criteria given in equation (5).

Figure 1 shows that the shear factor value was practically constant when the height was superior to the width of the cross section. The wooden beam was thus tested edgewise and the corresponding shear factor value was 5/6. However, when the height was less than the width, the K^* value decreased linearly with the base 10 logarithm of the h/e ratio. The corresponding K^* value associated with $h/e = 0.1$ was 0.823 instead of the standard 0.833. In this case, the wooden beam was tested flatwise and its mechanical behavior began to resemble that of a thick plate. Nevertheless, the relative error $|K^* - K| / K$ remained very low, with a maximum of 1.2%, when the width e was 10 times superior to the height h. Furthermore, this relative error was lower than the 6% bias between the exact and the simplified expression of K^*.

For a prismatic wooden beam in free–free transverse vibration, the eigenfrequencies can be deduced from equation (1) using an approximation of Taylor-Lagrange [3]. The solution given by Bordonné is thus presented [1]:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{E_XI_{Gx}}{\rho S}\frac{P_n}{L^4\left[1+QF_1(m)+Q\dfrac{E_X}{KG_{XY}}F_2(m)\right]}} \quad (9)$$

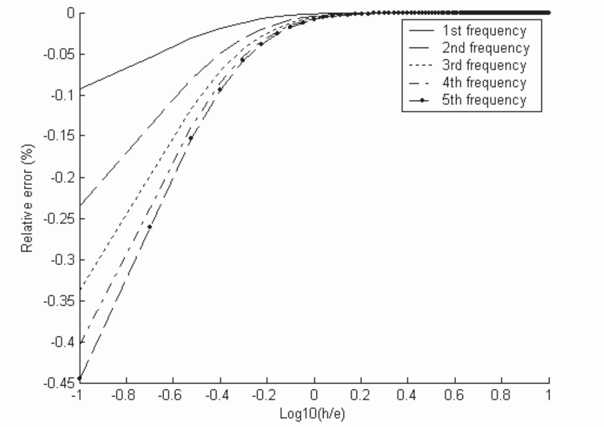

Figure 2. Variations in the relative error in vibration frequency calculations according to the base 10 logarithm of the *h*/*e* ratio.

where f_n is the vibration frequency of rank n, L the length of the beam, $Q = I_{Gz}/SL^2$, $F_1(m) = \theta^2(m) + 6\theta(m)$, $F_2(m) = \theta^2(m) - 2\theta(m)$, $\theta(m) = m\frac{\tan(m)\tanh(m)}{\tan(m) - \tanh(m)}$ and $m = \sqrt[4]{P_n} = (2n+1)\frac{\pi}{2}$, $n \in N^*$. The analytic formula of equation (9) was used with the shear factor K^* expression (8) to evaluate the size effect of the cross section on the first five vibration frequencies with a length-to-depth ratio of 10. Five frequencies were chosen since Bordonné's solution can only be applied for length-to-depth ratios above 10 with respect to the first five vibration modes [1]. Note also that for low length-to-depth ratio values the shear effect on the vibration frequencies is maximized and thus the size effect of the cross section is also maximized (9).

Figure 2 shows variations in the relative error for the first five vibration frequencies when comparing a calculation using the standard K value and a calculation using a K^* value based on the variation in the cross section height-to-width ratio (from 0.1 to 10). The following expression was applied to plot error variations on the basis of the above E_X, G_{XY} and ν_{XZ} values.

$$\frac{f_n^* - f_n}{f_n} = \sqrt{\frac{1 + QF_1(m) + Q\frac{E_X}{KG_{XY}}F_2(m)}{1 + QF_1(m) + Q\frac{E_X}{K^*G_{XY}}F_2(m)}} - 1\,. \tag{10}$$

The influence of variations in the cross section ratio on the first five vibration frequencies can be neglected due to the fact that their is a maximum error of –0.45% on the fifth frequency when the ratio *h*/*e* is equal to 0.1 (Fig. 2). Note that the influence of the cross section ratio increased with the frequency rank, in agreement with the fact that the shear influence increases with the rank. Equation (1) indeed implies that the particle motion resembles a pure shear motion with a velocity quasi-equal to $\sqrt{G_{XY}/\rho}$ when the frequency reaches infinity.

4. CONCLUSION

Modal analysis of free–free flexural vibrations provides a rapid and accurate means to determine the modulus of elasticity and the shear modulus of wooden beams. The shear modulus can thus be calculated using the shear factor that Timoshenko included in his specific model of vibrating beams. This model is an extension of the Euler-Bernoulli model because it takes the shear effect and rotary inertia into account. The combined effects of shear deformation and rotary inertia are indeed not negligible when the length to height ratio of the beam is within the 10 to 20 range.

In this case, the shear modulus can be accurately determined by using the shear factor. The aim of our study was to gain further insight into the effect of the cross section size on the shear factor value. We propose a new theoretical shear factor formulation for orthotropic material such as wood with a rectangular cross section – it allows determination of the shear modulus when the height to width ratio of the beam differs substantially from unity. We highlighted numerical trends for wood materials, which showed that:

- The shear factor was practically constant and equal to 5/6 when the height was superior to the width of the cross section.
- The shear factor decreased when the height was less than the width. However the bias between the calculated value and the constant value of 5/6 remained very low, with a maximum of 1.2% when the height to width ratio reached 0.1.
- The influence of cross section ratio variations on the first five vibration frequencies was negligible, with a maximum bias of –0.45% on the fifth frequency when the height to width ratio reached 0.1.

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